Exercise 10

A sample of tritium-3 decayed to 94.5% of its original amount after a year.

(a) What is the half-life of tritium-3?

(b) How long would it take the sample to decay to 20% of its original amount?

Solution

Part (a)

Assume that the rate of mass decay is proportional to the amount of mass remaining at any given time.

$$\frac{dm}{dt} \propto -m$$

There's a minus sign here because mass is being lost as time increases. Change the proportionality to an equation by introducing a (positive) constant k.

$$\frac{dm}{dt} = -km$$

Divide both sides by m.

$$\frac{1}{m}\frac{dm}{dt} = -k$$

Rewrite the left side by using the chain rule.

$$\frac{d}{dt}\ln m = -k$$

The function you have to differentiate to get $-k$ is $-kt + C$, where C is any constant.

$$\ln m = -kt + C$$

Exponentiate both sides.

$$e^{\ln m} = e^{-kt+C}$$

$$m(t) = e^C e^{-kt}$$

Use a new constant m_0 for e^C.

$$m(t) = m_0 e^{-kt} \tag{1}$$

Use the fact that the sample of tritium-3 decayed to 94.5% of its original amount after a year.

$$0.945 m_0 = m_0 e^{-k(1)}$$

$$0.945 = e^{-k}$$

$$\ln 0.945 = \ln e^{-k}$$

Solve for k.

$$\ln 0.945 = -k \ln e$$

$$k = -\ln 0.945 \approx 0.0565704 \text{ year}^{-1}$$

Equation (1) then becomes

$$\begin{aligned} m(t) &= m_0 e^{-(-\ln 0.945)t} \\ &= m_0 e^{\ln 0.945^t} \\ &= m_0 (0.945)^t. \end{aligned}$$

The half-life is defined as the amount of time it takes for a sample to decay to half its mass, so set $m(t) = m_0/2$ and solve the equation for t.

$$m(t) = \frac{m_0}{2}$$

$$m_0(0.945)^t = \frac{m_0}{2}$$

$$(0.945)^t = \frac{1}{2}$$

$$\ln(0.945)^t = \ln \frac{1}{2}$$

$$t \ln 0.945 = \ln 0.5$$

$$t = \frac{\ln 0.5}{\ln 0.945} \approx 12.2528 \text{ years}$$

Part (b)

To find how long it takes the sample to decay to 20% of its original mass, set $m(t) = 0.2m_0$ and solve the equation for t.

$$m(t) = 0.2m_0$$

$$m_0(0.945)^t = 0.2m_0$$

$$(0.945)^t = 0.2$$

$$\ln(0.945)^t = \ln 0.2$$

$$t \ln 0.945 = \ln 0.2$$

$$t = \frac{\ln 0.2}{\ln 0.945} \approx 28.4502 \text{ years}$$